A Generalized Framework for Stochastic Differential Equations with State-Dependent Nonlinearities

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Applied Mathematics
FST

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Outline

Motivation: Classical SDEs

A Foundational Tool

Stochastic Differential Equations (SDEs) are fundamental for modeling systems influenced by randomness.

Standard SDE form:

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t)$$

They are pivotal in:

- Mathematical Finance (e.g., asset pricing)
- Physics (e.g., particle diffusion)
- Biology (e.g., population dynamics)

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The Limitation of Standard SDEs

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The Goal:

Develop a more expressive SDE framework to capture these complex nonlinearities.

Objectives of This Work

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B. **Conduct** a rigorous theoretical analysis of the new model (well-posedness, properties).
C. **Design** and implement efficient numerical algorithms to simulate the model.
D. **Demonstrate** practical applications in finance, physics, and biology.

The Generalized Stochastic Model (GSM)

We introduce an SDE with state-dependent variable exponent functions $p_i(\cdot)$ in the drift and diffusion coefficients:

$$dX(t) = \underbrace{\mu\left(a_1 + b_1 X(t)^{p_1(X(t))}\right) X(t)^{\beta}}_{\text{drift}} \, dt + \underbrace{\sigma\left(a_2 + b_2 X(t)^{p_2(X(t))}\right)}_{\text{diffusion}} \, dW(t)$$

(**GSM**)

Model Components:

- $p_i(\cdot) : (0, \infty) \to (0, \infty)$
- $\mu, \sigma, a_i, b_i \in \mathbb{R}$
- $\beta \in \{0, 1\}$
- $W(t)$ is a standard Wiener process.

Key idea:

The exponents $p_1(X(t))$ and $p_2(X(t))$ allow the drift and diffusion terms to respond **nonlinearly** to the current state $X(t)$.

Note:

- $p(X(t)) = p(X(t, \omega))$ is a stochastic process.
- $p(X(t))$ is a not path-dependent, it depends only on the current value (state) of $X(t)$.
- $p(X(t))$ is measurable and adapted (to filtration $\{\mathcal{F}_t\}_{t \geq 0}$) if $X(t)$ is.

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Our research is structured in three phases:

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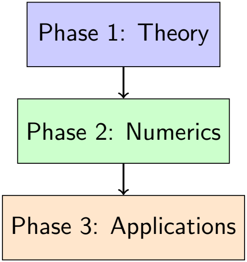
- **Phase 1: Theoretical Analysis**
  - Well-posedness of the GSM?
- **Phase 2: Numerical Development**
  - Implementation: How can we simulate this model?
- **Phase 3: Applications**
  - Validation: Does this model capture real-world data?



MSc Students: Skills and Tasks

Prospective MSc students will focus on the computational implementation and validation of the GSM:

- Be proficient in Python and its scientific computing Libraries (NumPy, SciPy, Pandas, Matplotlib/Seaborn).
- Documentation: LaTeX (Technical writing for thesis and project documentation).
- Stochastic Simulation: Monte Carlo method implementation, Random number generation and control, Variance reduction techniques, Statistical estimation.
- Numerical Methods for SDEs: Euler-Maruyama scheme implementation, Higher-order schemes (Milstein, etc.), Convergence analysis.
- Benchmarking against known special cases.
- Error analysis and stability tests.
- Train the GM with the historical data (e.g., S&P 500) to match real-world dynamics by calibrating the model parameters.

Outcomes of the MSc Study

By the end of the MSc Study:

- Students will have built robust numerical solvers for state-dependent stochastic models.
- Their work will produce computational validations supporting the theoretical findings of the project.
- They will have the opportunity for co-authorship on research papers and conference presentations.

I look forward to working with motivated and enthusiastic MSc students on this exciting computational research project.

Thank you!