

PROPERTIES OF LOGARITHMS

- *an invention of John Napier*
(1550 - 1617)

- *a reworking of the*
properties of exponentials a^y

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DEFINITION OF THE LOGARITHM

Given: $a > 0$ ($\neq 1$)

a strictly positive constant real number not equal to 1

and $x > 0$

any strictly positive real number.

Then

$\log_a x = y$ is defined to be the real number y such that:

y is the **exponent** for which $a^y = x$.

a is called the **base** of the logarithm.

x is called the **antilog to the base a of y** .

y is called the **log to the base a of x** .

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From the definition,

$\log_a x = y$ is defined to be the real number y such that:
 y is the **exponent** for which $a^y = x$.

We deduce that

$\log_a x = y$ is equivalent to $a^y = x$
provided $a > 0$ ($\neq 1$) and $x > 0$.

Observe the intricate connection between the **logarithm to the base a** and the **exponential with base a** .

EXPONENTIALS

Given: a any real number
and n a real number.

$$a^n = \frac{1}{a^{-n}} \quad a^0 = 1 \quad (a \neq 0)$$

$$* \text{ E1} \quad a^0 = 1 \quad ; \quad a^1 = a \quad (\text{Cases when } n = 0, 1)$$

$$* \text{ E2} \quad a^n \cdot a^m = a^{(n+m)}$$

$$* \text{ E3} \quad \frac{a^n}{a^m} = a^{(n-m)} \quad (a \neq 0)$$

$$* \text{ E4} \quad (a^n)^p = a^{(n \cdot p)}$$

$$* \text{ E5} \quad (a \cdot b)^n = a^n \cdot b^n \quad ; \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad a, b \text{ real} \\ b \neq 0$$

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Properties of Logarithms

- may be deduced from the corresponding properties of exponents because, by definition:

$$\log_a x = y \quad \text{is equivalent to} \quad a^y = x$$

$$\ast \text{ L1} \quad \log_a 1 = 0 \quad \text{is equivalent to} \quad a^0 = 1 \quad \ast \text{ E1}$$

log of 1 property

$$\ast \text{ L2} \quad \log_a a = 1 \quad \text{is equivalent to} \quad a^1 = a \quad \ast \text{ E1}$$

log of **base** property

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Evaluating using property *L1 of logarithms

EXAMPLE 1: Find $\log_{\frac{1}{2}} 1$.

Solution:

Check: $a = \frac{1}{2} > 0 \quad (\neq 1)$
 $x = 1 > 0$

By property

*** L1** $\log_a 1 = 0$ log of 1 property

Answer: $\log_{\frac{1}{2}} 1 = 0$

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Evaluating using property *L2 of logarithms

EXAMPLE 2: Find $\log_{45} 45$.

Solution:

Check: $a = 45 > 0 \quad (\neq 1)$

$$x = 45 > 0$$

By property

*** L2**

$$\log_a a = 1$$

log of **base** property

Answer: $\log_{45} 45 = 1$

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Properties of Logarithms

- may be deduced from the corresponding properties of exponents because, by definition:

$$\log_a x = y \quad \text{is equivalent to} \quad a^y = x$$

*** L3** $a^{\left(\log_a x\right)} = x$ is equivalent to $a^y = x$
by definition of y

exponential of log property

*** L4** $\log_a \left(a^y\right) = y$ is equivalent to $a^y = a^y$
by definition of logarithm

log of exponential property

OR since $a^y = x$

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Evaluating using property *L3 of logarithms

EXAMPLE 3: Evaluate $10^{\left(\log_{10} 7654\right)}$.

Solution:

Check: $a = 10 > 0 \quad (\neq 1)$

$x = 7654 > 0$

By property

*** L3** $a^{\left(\log_a x\right)} = x$ exponential of a **same base** log property

Answer: $10^{\left(\log_{10} 7654\right)} = 7654$

Note that the answer is an antilog.

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Evaluating using property *L4 of logarithms

EXAMPLE 4: Evaluate $\log_e \left(e^{\sqrt{2}} \right)$.

Solution:

Check: $a = e \sim 2.71828 > 0 \quad (\neq 1)$

$$x = e^{\sqrt{2}} > 0$$

By property

*** L4** $\log_a \left(a^y \right) = y$ log of a **same base** exponential property

Answer: $\log_e \left(e^{\sqrt{2}} \right) = \sqrt{2}$

log to the base $e \sim 2.71828$
is called the **natural logarithm**

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Properties of Logarithms

- may be deduced from the corresponding properties of exponents because, by definition:

$$\log_a x = y \quad \text{is equivalent to} \quad a^y = x$$

*** L5** $\log_a (n \cdot m) = \log_a n + \log_a m$

log of a **product** property

$$\begin{aligned} \text{Let } \log_a n &= y_n & \left(a^{y_n} = n \right) \\ \text{and } \log_a m &= y_m & \left(a^{y_m} = m \right) \end{aligned}$$

$$\log_a (n \cdot m) = \log_a n + \log_a m = (y_n + y_m)$$

$$\text{is equivalent to } a^{(y_n + y_m)} = n \cdot m$$

$$\text{or } a^{(y_n + y_m)} = a^{y_n} \cdot a^{y_m} \quad \textbf{* E2}$$

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Evaluating using property *L5 of logarithms

EXAMPLE 5: Evaluate $\log(80)$.

Solution:

Check: $a = 10 > 0 \quad (\neq 1)$

$x = 80 > 0$

If the base is not specified,
it is assumed to be $a = 10$.

Factor the antilog and then use the following property
to split the factors into two separate logarithms.

$$\text{* L5 } \log_a(n \cdot m) = \log_a n + \log_a m$$

log of a **product** property

$$\log_{10} 80 = \log_{10} (8 \cdot 10) = \log_{10} 8 + \log_{10} 10$$

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Evaluating using property *L5 of logarithms

EXAMPLE 5: Evaluate $\log(80)$.

Solution (continued):

$$\log_{10} 80 = \log_{10} (8 \cdot 10) = \log_{10} 8 + \log_{10} 10$$

using a calculator

by property * **L2**
log of the **base**
property

Answer: $\log(80) = 1.90309$ to 5 decimal places

Note that:

$$10^1 = 10 < 80 < 10^2 = 100$$

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Properties of Logarithms

- may be deduced from the corresponding properties of exponents because, by definition:

$$\log_a x = y \quad \text{is equivalent to} \quad a^y = x$$

*** L6** $\log_a \left(\frac{n}{m} \right) = \log_a n - \log_a m$

log of a **quotient** property

$$\begin{aligned} \text{Let } \log_a n &= y_n & \left(a^{y_n} = n \right) \\ \text{and } \log_a m &= y_m & \left(a^{y_m} = m \right) \end{aligned}$$

$$\log_a \left(\frac{n}{m} \right) = \log_a n - \log_a m = (y_n - y_m)$$

$$\text{is equivalent to } a^{(y_n - y_m)} = \frac{n}{m} = \frac{a^{y_n}}{a^{y_m}} \quad \textbf{* E3}$$

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Evaluating using property *L6 of logarithms

EXAMPLE 6: Evaluate $\log_3 \left(\frac{1}{729} \right)$.

Solution:

Check: $a = 3 > 0 \quad (\neq 1)$

$$x = \frac{1}{729} > 0$$

Use the following property to split the numerator and the denominator of the antilog into two separate logarithms.

$$\text{* L6 } \log_a \left(\frac{n}{m} \right) = \log_a n - \log_a m$$

log of a **quotient** property

$$\log_3 \left(\frac{1}{729} \right) = \log_3 1 - \log_3 729$$

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Evaluating using property *L6 of logarithms

EXAMPLE 6: Evaluate $\log_3 \left(\frac{1}{729} \right)$.

Solution (continued):

$$\log_3 \left(\frac{1}{729} \right) = \log_3 1 - \log_3 729$$

by property $\ast$ L1
log of 1 property

since $3^6 = 729$

Answer: $\log_3 \left(\frac{1}{729} \right) = -6$

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Properties of Logarithms

- may be deduced from the corresponding properties of exponents because:

$$\log_a x = y \quad \text{is equivalent to} \quad a^y = x$$

*** L7** $\log_a (n^r) = r \cdot \log_a n = r \cdot y_n$
log of a **power** property

is equivalent to $a^{(r \cdot y_n)} = n^r$
or $a^{(r \cdot y_n)} = (a^{y_n})^r$ *** E4**

In this case, we set $\log_a n = y_n$ $\left(a^{y_n} = n \right)$

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Evaluating using property *L7 of logarithms

EXAMPLE 7: Evaluate $\log_2(\sqrt{2})$.

Solution:

Check: $a = 2 > 0 \ (\neq 1)$
 $x = \sqrt{2} > 0$

Recalling that the square root is really the power $1/2$,
we may use the property:

$$\text{* L7} \quad \log_a(n^r) = r \cdot \log_a n$$

$$\log_2(\sqrt{2}) = \log_2(2^{1/2}) = \frac{1}{2} \cdot \log_2 2 = \frac{1}{2} \cdot 1$$

Answer: $\log_2(\sqrt{2}) = \frac{1}{2}$

by property *L2
log of the **base**
property

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Properties of Logarithms

- may be deduced from the corresponding properties of exponents because:

$$\log_a x = y \quad \text{is equivalent to} \quad a^y = x$$

* L8 Change of Base Formula

- used to find the logarithm of a number x to the base a in terms of another base b :

$$\log_a x = \frac{\log_b x}{\log_b a}$$

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Evaluating using property *L8 of logarithms

EXAMPLE 8: Evaluate $\log_8(\sqrt{2})$.

Solution: **Check:** $a = 8 > 0$ ($\neq 1$)
 $x = \sqrt{2} = 2^{1/2} > 0$

By changing the base to $b = 2$, calculations may be easier.

*** L8** $\log_a x = \frac{\log_b x}{\log_b a}$ **Change of Base Formula**
 from base $a = 8$ to base $b = 2$

$$\log_8(2^{1/2}) = \frac{\log_2(2^{1/2})}{\log_2(8)} = \frac{1/2 \cdot \log_2 2}{3} = \frac{1/2}{3}$$

Answer: $\log_8(\sqrt{2}) = 1/6$

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Properties of Logarithms

* L1 $\log_a 1 = 0$
log of 1

* L2 $\log_a a = 1$
log of the base

* L3 $a^{\log_a x} = x$
exponential of log

* L4 $\log_a (a^y) = y$
log of exponential

* L5 $\log_a (n \cdot m) = \log_a n + \log_a m$
log of a product

* L6 $\log_a \left(\frac{n}{m} \right) = \log_a n - \log_a m$
log of a quotient

* L7 $\log_a (n^r) = r \cdot \log_a n$
log of a power

* L8 $\log_a x = \frac{\log_b x}{\log_b a}$

Change of Base Formula
from base **a** to base **b**

