

THE DEFINITION OF LOGARITHMS to VARIOUS BASES

- *an invention of John Napier*
(1550 - 1617)

--> *a reworking of the exponential equation*

$$a^y = x$$

LOGARITHMS

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The logarithm was intended to make certain arithmetical calculations easier. In fact, through the use of logarithmic tables, the process of long multiplication of large numbers can be reduced to long addition, a much simpler process.

DEFINITION of the LOGARITHM to the BASE a

Given: $a > 0$ ($\neq 1$)

a strictly positive constant real number not equal to 1

and: $x > 0$

any strictly positive real number.

Then: $\log_a x = y$ is defined to be the number y such that:

y is the **exponent** for which $a^y = x$.

Observe the intricate connection between the **logarithm to the base a** and the **exponential with base a** .

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DEFINITION OF THE LOGARITHM

Given: $a > 0$ ($\neq 1$)

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and $x > 0$

any strictly positive real number.

Then

$\log_a x = y$ is defined to be the real number y such that:

y is the **exponent** for which $a^y = x$.

a is called the **base** of the logarithm.

x is called the **antilog to the base a of y** .

y is called the **log to the base a of x** .

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Evaluating logarithms using the definition

EXAMPLE 1: Find $\log_{10} 10000$.

Check: $a = 10 > 0$ ($\neq 1$)

$$x = 10000 > 0$$

Solution:

$\log_a x = y$ is defined to be the **exponent** y
for which $a^y = x$.

$\log_{10} 10000$ is defined to be the **exponent** y
for which $10^y = 10000$.

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Evaluating logarithms using the definition

EXAMPLE 1: Find $\log_{10} 10000$.

Solution (continued):

$\log_{10} 10000$ is defined to be the exponent y
for which $10^y = 10000$.

Begin by experimenting with various powers y of 10 .

$y = 1$	$10^y = 10^1 = 10$
$y = 3$	$10^y = 10^3 = 1000$
$y = 4$	$10^y = 10^4 = 10000$

Answer: $\log_{10} 10000 = 4$

log to the base 10
is called the **common logarithm**

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Evaluating logarithms using the definition

EXAMPLE 2: Find $\log_5 3125$.

Check: $a = 5 > 0$ ($\neq 1$)

$$x = 3125 > 0$$

Solution:

$\log_a x = y$ is defined to be the exponent y
for which $a^y = x$.

$\log_5 3125$ is defined to be the exponent y
for which $5^y = 3125$.

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Evaluating logarithms using the definition

EXAMPLE 2: Find $\log_5 3125$.

Solution (continued):

$\log_5 3125$ is defined to be the exponent y
for which $5^y = 3125$.

Begin by experimenting with various powers y of 5 .

$$y = 4 \qquad 5^y = 5^4 = 25 \cdot 25 = 625$$

$$y = 5 \qquad 5^y = 5^5 = 625 \cdot 5 = 3125$$

Answer: $\log_5 3125 = 5$

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Evaluating logarithms using the definition

EXAMPLE 3: Find $\log_2 \left(\frac{1}{4} \right)$.

Check: $a = 2 > 0 \quad (\neq 1)$

$$x = \frac{1}{4} > 0$$

Solution:

$\log_2 \left(\frac{1}{4} \right)$ is defined to be the exponent y
for which $2^y = \frac{1}{4}$.

If we substitute with various positive integral powers y of 2 ,
 $y = 1, 2, 3, \dots$ then 2^y evaluates to an integer > 2 .

$$y = 0 \qquad 2^y = 2^0 = 1$$

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Evaluating logarithms using the definition

EXAMPLE 3: Find $\log_2 \left(\frac{1}{4} \right)$.

Solution (continued):

Try experimenting with negative integral powers y of 2 .

$$y = -1 \qquad 2^y = 2^{-1} = \frac{1}{2^{+1}} = \frac{1}{2}$$

$$y = -2 \qquad 2^y = 2^{-2} = \frac{1}{2^{+2}} = \frac{1}{4}$$

Answer: $\log_2 \left(\frac{1}{4} \right) = -2$

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Evaluating logarithms using the definition

EXAMPLE 4: Find $\log_{253} 1$.

Check: $a = 253 > 0$ ($\neq 1$)

$$x = 1 > 0$$

Solution:

$\log_{253} 1$ is defined to be the exponent y
for which $253^y = 1$.

From the Laws of Exponents, we know that $a^0 = 1$
for any positive base $a \neq 1$.

Answer: $\log_{253} 1 = 0$

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Evaluating logarithms using the definition

EXAMPLE 5: Find $\log_{-7} 49$.

Check: $a = -7 < 0$

$$x = 49 > 0$$

Solution:

$\log_{-7} 49 = y$ is not defined despite that fact
that $(-7)^2 = 49$,

since the base of the logarithm $a = -7 < 0$
violates the condition in the definition that it be a
strictly positive constant.

Answer: $\log_{-7} 49$ is undefined.

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Evaluating logarithms using the definition

EXAMPLE 6: Find $\log_{1000}(-1000)$.

Check: $a = 1000 > 0$ ($\neq 1$)

$x = -1000 < 0$

Solution:

$\log_{1000}(-1000)$ is not defined

because the **antilog** $x = (-1000) < 0$
violates the condition in the definition that it be a
strictly positive real number.

Note that: $\log_{1000}(-1000) \neq -1$

Answer: $\log_{1000}(-1000)$ is undefined.

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Evaluating logarithms using the definition

EXAMPLE 7: Find $\log_2 6$.

Check: $a = 2 > 0 \quad (\neq 1)$
 $x = 6 > 0$

Solution:

$\log_2 6$ is defined to be the exponent y
for which $2^y = 6$.

By experimenting with various powers y of 2 , we know that:

$$2^2 = 4 < 6 < 2^3 = 8$$

$$2^2 < 2^y < 2^3$$

$$2 < y < 3$$

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Evaluating logarithms using the definition

EXAMPLE 7: Find $\log_2 6$.

Solution (continued):

$\log_2 6$ is defined to be the exponent
for which $2^y = 6$.

$$2 < y < 3$$

Without a scientific calculator or a set of log tables, we will not be able to find the, even approximate, answer.

Therefore, use a calculator to evaluate this logarithm because the answer is not an integer.

Answer: $\log_2 6 = 2.5849625007$
to 10 decimal places .

Is this the first time you have seen an exponent which is a decimal?

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Summary

meaning that

$$\log_{10} 10000 = 4$$

$$10^4 = 10000$$

$$\log_5 3125 = 5$$

$$5^5 = 3125$$

$$\log_{253} 1 = 0$$

$$253^0 = 1$$

$$\log_2 \left(\frac{1}{4} \right) = -2$$

$$2^{(-2)} = \frac{1}{4}$$

$$\log_2 6 = 2.5849625007$$

$$2^{2.5849625007} = 6$$

$\log_{-7} 49$ is undefined since the base $a = -7 < 0$.

$\log_{1000} (-1000)$ is undefined since $x = -1000 < 0$.

antiLOGARITHMS

- *another way of looking at
exponentials*

Evaluating antilogarithms using the definition

EXAMPLE 8: Find the antilog of 3 to the base 10.

Solution: Recall the definition of logarithm.

$\log_a x = y$ is defined to be the real number y such that:
 y is the exponent for which $a^y = x$.

a is called the **base** of the logarithm.

x is called the **antilog to the base a of y** .

y is called the **log to the base a of x** .

In the problem, we are looking for the antilog x .

We are given the base $a = 10$ and the exponent $y = 3$.

antiLOGARITHMS

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exponentials*

Evaluating antilogarithms using the definition

EXAMPLE 8: Find the antilog of 3 to the base 10 .

Solution (continued):

Therefore, the antilog of 3 to the base 10
is $x = a^y = 10^3 = 1000$.

Answer: $\text{antilog}_{10} 3 = 1000$

An equivalent statement is: $\log_{10} 1000 = 3$

antiLOGARITHMS

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Evaluating antilogarithms using the definition

EXAMPLE 8: Find the antilog of 3 to the base 10 .

Answer: $\text{antilog}_{10} 3 = 1000$

An equivalent statement is: $\log_{10} 1000 = 3$

Observation: Apply **antilog to the base 10** to both sides of the above equation.

$$\text{antilog}_{10} (\log_{10} 1000) = \text{antilog}_{10} 3$$

From the answer to **Example 8**, it looks as if the antilog applied to the log of the same base 10 “cancel each other out” and leave only the number 1000 .

This, in effect, what actually happens because the logarithm and the antilogarithm are said to be inverse functions.

antiLOGARITHMS

- *another way of looking at
exponentials*

Evaluating antilogarithms using the definition

EXAMPLE 9: Find the antilog of 7 to the base 2 .

Solution:

In the problem, we are looking for the antilog x .

We are given the base $a = 2$ and the exponent $y = 7$.

Therefore, the antilog of 7 to the base 2
is $x = a^y = 2^7 = 128$.

Answer: $\text{antilog}_2 7 = 128$

An equivalent statement is: $\log_2 128 = 7$

