THE MATHEMATICS of TENSORS – an Introduction

(an addendum to The General Theory of Relativity – an Introduction) **by Julie Peschke, Athabasca University**

In what follows, it is important to remember that the points or vectors associated with 2- or 3-dimensional objects or even higher dimensional spaces may be expressed in terms of a variety of coordinate systems.

When we go about our daily lives, at any particular location on the earth's surface, we do not observe that the streets are moving. That is because we are moving with them at the same uniform rate of travel (unless, of course, there is an earthquake at our location or we are in the throes of a hurricane). A person in another city two hundred miles away would be seeing another grid of street patterns but still would not detect that that grid is also moving. However, an observer in outer space would be able to detect both movements from a space station, including his/her own motion in the earth's orbit. This is what physicists would call three different frames of reference (or location perspectives) concerning the earth's movement in the context of Einstein's space-time continuum model.

In the General Theory of Relativity, the laws of moving bodies from one state to another must be independent of the frame of reference or the perspective of the observer from the locale in which s(he) is positioned within a given coordinate system. This is done to avoid bias in the scientific observations and any ensuing errors in the mathematics which may occur by preferring one frame of reference to another. In other words, the transformations and dynamics of moving objects must not be dependent upon or associated with a particular coordinate system stamped on a particular space. In fact, different observers measure different distances and different times depending on their locations and velocities – sometimes leading to distance contraction or time dilation, however small. Remember that, even in our every-day world, time = distance/speed is a ratio and, therefore, a relative measure.

Moreover, now suppose that you are walking along a narrow, straight, smoothly paved, lane in a city. You think that you are walking along a straight path but you're not. In fact, in terms of the surrounding 3-dimensional ambient space, your actual path is slightly curved due to both the centrifugal and the coriolis forces exerted on the earth's surface as is rotates, not to mention the change in the position of the earth in its orbit over that time period. In addition, depending on your precise location on the earth's surface in terms of latitude and longitude, these inertial forces may differ in magnitude and direction. (This should remind you of a mathematical object called a "vector.")

These phenomena may be thought of as a 4-dimensional space-time parallax.

In order to account for this kind of experiential parallax, tensors became the mathematical object of choice for the theoretical model because they can be expressed independently from the frame of reference and can account for movement from one locale to another. They are often considered as the generalization of a vector. However, they may also be conceived of as a generalization of the concept of a transformation on a space of vectors with an associated matrix of changes to the base vectors. The only stipulation is that tensors must keep the 0-vector fixed from one space to another. Essentially a tensor maps a flat space of some dimension into another flat space. Note that tensors of different ranks may be defined on vectors in an arbitrary number of dimensions, $\mathbb{R}^n$. In addition, movement in a space, curved or not, is calculated using vectors and derivatives in the flat spaces that approximate (to the closest possible degree) the changing locales of the moving point – namely the associated tangent planes to the path of motion.

In differential geometry, an *intrinsic* geometric statement about a surface can be described by a tensor map on the tangent planes to that surface, and then doesn't need to (though in some cases may) make reference to coordinate systems.

For a more intuitive perspective of what tensors are, see **Tensors Explained Intuitively: Covariant, Contravariant, Rank** (published on July 20, 2017 by Physics Videos by Eugene Khutoryansky). Note that, in this video, all of the tensors discussed act on vectors in the 3-dimensional space, $\mathbb{R}^3$. In the vernacular, it may be said that tensor components of various ranks "cover all the combinations of bases" much like hitters in a baseball game who actually get to first base.

Tensor of rank 0 acting on an *n*-dimensional space of vectors, $\mathbb{R}^n$, results in a scalar (a number) which has magnitude but NO direction. Its action associates a number with **every vector** in the space.

An example of such a tensor is the one which assigns to every vector, its length which is a single numerical entry. In the notation below, the lower (x) indicates the frame of reference having an *n*-dimensional $x_1x_2\ldots x_n-$coordinate system and the upper indices $V^i, 1\le i\le n,$ represent the contravariant components of the vector V in that frame of reference.

$$T:\mathbb{R}^n\to\mathbb{R}^0=\mathbb{R}$$

$$V_{(x)}=\begin{bmatrix} V^1{}_{(x)} \\ V^2{}_{(x)} \\ \ldots \\ V^n{}_{(x)} \end{bmatrix}\mapsto \| V_{(x)}\| = \sqrt{(V^1{}_{(x)})^2+(V^2{}_{(x)})^2+\ldots+(V^n{}_{(x)})^2}$$

Tensor of rank 1 acting on a 3-dimensional space of vectors results in a vector which has magnitude AND direction. It associates a number with **each single basis vector** (hence, rank 1) in the 3-dimensional ambient vector space $\mathbb{R}^3$. Recall that a basis is a minimal set of vectors pointed in independent directions (in this case 3) such that all other vectors in the space may be expressed as linear combinations of the basis vectors using scalar (number) multiplication and addition.

Example: Suppose that the tenor T associates the numbers 1, 2, and -3, respectively, to the three basis vectors in any frame of $\mathbb{R}^3$. The transformation matrix of the tenor T on $\mathbb{R}^3$ will be the (3 X 3) matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, noting that, in this case, the matrix entries do not represent the coordinates of the basis vectors – only how the tensor transforms them. Thus the tenor T will map a vector to its coordinates/components in that frame modified by those values. The resultant vector may be represented as a (3 x 1) column matrix with 3 entries, which is just another vector in $\mathbb{R}^3$.

$$T:\mathbb{R}^3 \to \mathbb{R}^3$$

$$V = \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \bullet \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} = \begin{bmatrix} 1 \cdot V^1 \\ 2 \cdot V^2 \\ -3 \cdot V^3 \end{bmatrix} = T(V)$$

The multiplication above is matrix multiplication, that is the product of each matrix row by the vector column, entry by entry and then added together. In this case, a single column matrix can be interpreted as a the components of a vector in the given frame of basis vectors.

Tensors of rank 2 determine a relationship between vectors in two different frames of reference. Mathematically, it may be represented as a transformation from a flat space of some given dimension into another flat space of the same dimension which keeps the 0-vector fixed.

Example 1: a tensor of rank 2 of type (1-covariant, 1-contravariant) acting on $\mathbb{R}^2$

Tensors of rank 2 acting on a 2-dimensional space would be represented by a 2 x 2 matrix with $4 = 2^2$ components associated with all possible pairings in 2 dimensions, namely 11, 12, 21, 22.

$$T:\mathbb{R}^2 \to \mathbb{R}^2$$

$$V_{(x)} \mapsto V_{(y)}$$

$$T_{matrix} = \begin{bmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{bmatrix}$$

$$\begin{bmatrix} V^1{}_{(x)} \\ V^2{}_{(x)} \end{bmatrix} \mapsto \begin{bmatrix} V^1{}_{(y)} \\ V^2{}_{(y)} \end{bmatrix} = \begin{bmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{bmatrix} \bullet \begin{bmatrix} V^1{}_{(x)} \\ V^2{}_{(x)} \end{bmatrix} = \begin{bmatrix} T_1^1 \cdot V^1{}_{(x)} + T_2^1 \cdot V^2{}_{(x)} \\ T_1^2 \cdot V^1{}_{(x)} + T_2^2 \cdot V^2{}_{(x)} \end{bmatrix}$$

In this case, the upper index (the contravariant part) is one of two vector components in a flat 2-dimensional tangent plane. The lower index (the covariant part) represents one of the two basis vectors in the same flat 2-dimensional tangent plane. The above tensor T is a 1-covariant, 1-contravariant object, or a rank 2 tensor of type (1, 1) on $\mathbb{R}^2$.

Example 2: a tensor of rank 2 of type (1-covariant, 1-contravariant) acting on $\mathbb{R}^3$

Tensors of rank 2 acting on a 3-dimensional space would be represented by a 3 x 3 matrix with $9 = 3^2$ components associated with possible **pairings** in 3 dimensions, namely 11, 12, 13, 21, 22, 23, 31, 32, 33.

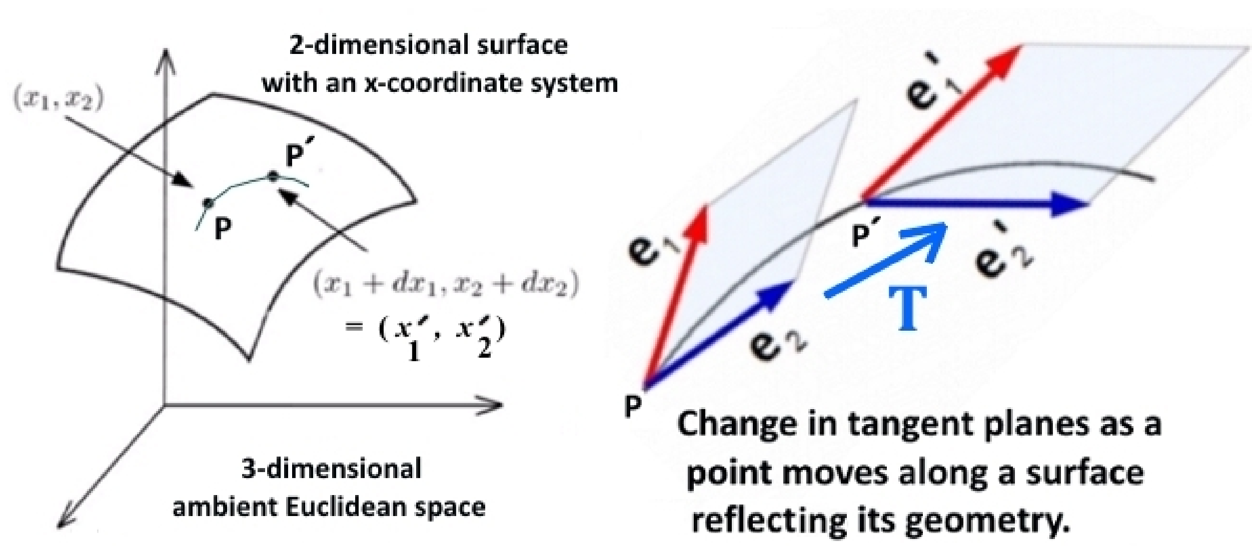
$$T:\mathbb{R}^3 \to \mathbb{R}^3 \qquad T_{matrix} = \begin{bmatrix} T_1^1 & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 \end{bmatrix}$$

$$V_{(x)} = \begin{bmatrix} V^1{}_{(x)} \\ V^2{}_{(x)} \\ V^3{}_{(x)} \end{bmatrix} \mapsto V_{(y)} = \begin{bmatrix} V^1{}_{(y)} \\ V^2{}_{(y)} \\ V^3{}_{(y)} \end{bmatrix} = \begin{bmatrix} T_1^1 & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 \end{bmatrix} \bullet \begin{bmatrix} V^1{}_{(x)} \\ V^2{}_{(x)} \\ V^3{}_{(x)} \end{bmatrix} = \begin{bmatrix} T_1^1 \cdot V^1{}_{(x)} + T_2^1 \cdot V^2{}_{(x)} + T_3^1 \cdot V^3{}_{(x)} \\ T_1^2 \cdot V^1{}_{(x)} + T_2^2 \cdot V^2{}_{(x)} + T_3^2 \cdot V^3{}_{(x)} \\ T_1^3 \cdot V^1{}_{(x)} + T_2^3 \cdot V^2{}_{(x)} + T_3^3 \cdot V^3{}_{(x)} \end{bmatrix} = T(V)$$

In this case, the upper index is one of three vector components in a flat 3-dimensional tangent plane. The lower index represents one of three basis vectors in the same flat 3-dimensional tangent plane. The above tensor T is a 1-covariant, 1-contravariant object, or a rank 2 tensor of type (1, 1) on $\mathbb{R}^3$.

Example 3: a tensor of rank 2 of type (2-covariant, 0-contravariant) which is called a covariant tensor of rank 2

The components of this type of tensor, the $T_{i\,j}$, are not expressed in terms of the individual vector components, only the transformations of the basis vectors (hence the nomenclature 'covariant' meaning varying in sync with the basis vectors of the space). However the matrix of tensor components transforms vectors in one flat space to another flat space of the same dimension. See the diagram below.



$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$V_{(x)} \mapsto V_{(x')}$$
$$\begin{bmatrix} V^1{}_{(x)} \\ V^2{}_{(x)} \end{bmatrix} \mapsto \begin{bmatrix} V^1{}_{(x')} \\ V^2{}_{(x')} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \bullet \begin{bmatrix} V^1{}_{(x)} \\ V^2{}_{(x)} \end{bmatrix}$$

$$T_{matrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

In the $T_{i\,j}, 1 \le i, j \le 2$, the two lower indices represent the tensor components for all pairs of two basis vectors in the tangent plane. The transformation of the components of an arbitrary vector V is defined in terms of the tensor components.

$$\begin{bmatrix} V^1{}_{(x')} \\ V^2{}_{(x')} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \bullet \begin{bmatrix} V^1{}_{(x)} \\ V^2{}_{(x)} \end{bmatrix} = \begin{bmatrix} T_{11} \cdot V^1{}_{(x)} + T_{12} \cdot V^2{}_{(x)} \\ T_{21} \cdot V^1{}_{(x)} + T_{22} \cdot V^2{}_{(x)} \end{bmatrix} = \begin{bmatrix} \sum_i T_{1i} \cdot V^i{}_{(x)} \\ \sum_i T_{2i} \cdot V^i{}_{(x)} \end{bmatrix}$$ defined using matrix multiplication.

The four tensor components (matrix entries) define the change in the orientation and position of the tangent planes at each point as a point moves along a curve in the 2-dimensional surface embedded in 3-dimensional Euclidean space.

Tensors of rank 3 or greater may be defined on spaces of higher dimensions. Their representations of components would consists of stacks of matrices. Because Einstein's field equations only feature tensors of rank 2, such higher rank tensors will not be discussed here.

How specific rank 2 covariant tensors in Einstein's space-time model, namely (R_{uv}, g_{uv}, T_{uv}) , transform a space of vectors will depend on their geometric purpose and will not be discussed here. However, the mathematical minutia of details may be found in the following videos.

THE MATHEMATICAL FOUNDATIONS of EINSTEIN'S FIELD EQUATIONS

For more information on how distance along a path on a curved surface and curvature of a surface at a specified point on it are defined and calculated, see the following sequence of videos.

What are Contrvariant and Covariant Components of a Vector? (published on May 20, 2009 by Mathview)
0:00 – 5:21 General discussion on the applications of differential geometry
5:22 – end The mathematics of contravariant and covariant components of a vector

Vectors as Directional Derivatives (published on March 27, 2017 by Robert Davie)

Basis Vectors and the Metric Tensor (published on December 22, 2016 by Robert Davie)

Parallel Transport & Curvature (published on February 11, 2015 by The WE-Heraeus International Winter School on Gravity and Light)
This video discusses parallel transport which is a way of measuring the change in direction of an object on a surface as it moves around on that surface in a parallel way in the higher dimensional ambient space. It uses the directional derivative of a vector or the covariant derivative of a tensor along a *closed curve* on the surface. If there is no bending on the surface, there is no change in direction of the vector representing the object. Consequently, the notion of parallel transport can be used to measure the curvature of a surface in the ambient higher dimensional space.

The Metric Tensor g_{uv} : is a covariant tensor of rank 2 on a space of vectors. It provides a way to measure distance or length on a curved surface. It is a generalization of the Euclidean metric on flat space which is based on Pythagorum's Theorem of the lengths of the sides of a right triangle.

Recall that the Euclidean metric measures the distance squared between two points in an arbitrary *n*-dimensional flat space and is derived from Pythagorum's Theorem in a 2-dimensional plane:
$ds^2 = dx^2 + dy^2$ where ds is the distance between the points (the length of the hypotenuse of the right triangle associated with the points), dx is the difference in the x-coordinates and dy is the difference in the y coordinates of the points (the respective lengths of the sides of the triangle).

Relativity 7a – differential geometry I (published on December 23, 2011 by viascience) – a look at the mathematical framework for the General Theory of Relativity and how the metric tensor plays a major role in its formulation.

Relativity 7b – differential geometry II (0:00 – 8:17 minutes) (published on January 22, 2012 by viascience) – a quick overview of the metric tensor on a curved surface in an arbitrary number of dimensions.

Riemannian Curvature Tensor

Relativity 7b – differential geometry II (8:17 – 11:28 minutes) (published on January 22, 2012 by viascience) – a quick overview as to how one can tell if a surface is curved, including a discussion of the Riemannian curvature tensor expressed in terms of the Christoffel symbols.

Riemannian Curvature Tensor (published on April 19, 2017 by David Butler) – includes a discussion on the mathematical components involved in calculating the curvature of a surface at a point on the surface, including the notion of a geodesic of a space (the straight line counterpart or path of shortest distance between two points on that surface) and the Ricci Curvature Tensor.

The Scalar Curvature R: sometimes called the Ricci curvature scalar, assigns a number to every point on a surface embedded in flat Euclidean space, $\mathbb{R}^n$, which measures the curvature of the surface at that point. The curvature value is found by comparing the volume of a *geodesic* ball about the point on the surface to the volume of a corresponding ordinary ball of radius 1 in a flat Euclidean space. If there is no deviation in the compared values, the surface has curvature 0 at that point because flatness of a space implies curvature 0.

The Ricci Curvature Tensor R_{uv}: is used to measure *how* the volume of an object (in particular, a small wedge of a geodesic ball) about a point on a curved surface deviates from that of the standard ball in flat Euclidean space. As such, it provides one way of measuring the degree to which the geometry determined by a given metric tensor might differ from the geometry of the ordinary flat Euclidean *n*-space metric. As a tensor (that is a linear map), it is able to track the changes in curvature (including the various **contours** of a surface) as a point moves across the surface.

> **Relativity 7b – differential geometry II** (11:28 – end, in minutes) (published on January 22, 2012 by viascience)
>
> **Ricci Tensor and Scalar** (published on June 20, 2016 by Robert Davie) – a look at the derivation of both the Ricci curvature tensor and the Ricci curvature scalar using the symmetry properties of the Riemannian curvature tensor.

Einstein himself, after proposing his revolutionary theory of general relativity, came to some startling conclusions:

> "In the General Theory of Relativity, space and time cannot be defined in such a way that differences of the spatial coordinates can be directly measured by the unit measuring-rod, or differences in the time co-ordinate by a standard clock. The method hitherto employed for laying co-ordinates into the space-time continuum in a definite manner thus breaks down ... "
> (Einstein, 1916, *The Foundation of the General Theory of Relativity*)

The Principle of Relativity

> **Relativity 8 – the yardstick of spacetime** (published on February 18, 2012 by viascience)

The Stress Energy Momentum Tensor T_{uv}: an attribute of matter, radiation, and non-gravitational force fields, it describes how much energy and momentum a moving body has at each point on it. In Einstein's mathematical model, it indicates the density of energy and momentum at each point in space time and dictates how space-time curves.

> **Energy-momentum tensor** (published on February 24, 2017 by dXoverdteqprogress)
>
> **The stress tensor** (published on October 16, 2014 by Brian Storey)

The Cosmological Constant Λ: Modern field theory associates this term with the energy density of a vacuum. It governs the rate at which the expanding universe is accelerating and is associated with the notion of dark energy. Einstein introduced this constant into his field equations in order extend the forces of gravity to 'push,' thus leading to acceleration and repulsion from other objects, not just 'pull' generating an attraction towards an object.

The Universal Gravitational Constant G: is the constant appearing in Newton's Law of Gravitation. It is related to the gravitational force of attraction between two bodies and is equal to 6.67408313131 x 10^{-11} N m^2/kg^2 , where N is measured in units of newton force. This constant is not the same as the constant gravity g which denotes the acceleration due to gravity.

The Universal Constant c : the speed of light in a vacuum equal to 299,792,458 meters/second. This is the fastest any object can move in our cosmos.

Implications of General Relativity

- **Relativity 10a – uniform gravity / acceleration I** (published on May 15, 2012 by viascience)
- **Relativity 10b – uniform gravity / acceleration II** (published on June 7, 2012 by viascience)
- **Relativity 11a – spherical bodies and black holes I** (published on July 16, 2012 by viascience)
- **Relativity 11b – spherical bodies and black holes II** (published on March 14, 2013 by viascience)
- **Relativity 11c – spherical bodies and black holes III** (published on January 18, 2016 by viascience)
- **Relativity 11d – spherical bodies and black holes IV** (published on April 25, 2016 by viascience)
- **Relativity 11e – spherical bodies and black holes V** (published on April 26, 2016 by viascience)
- **Relativity 11f – spherical bodies and black holes VI** (published on December 12, 2016 by viascience)

Einstein's Field Equations – for beginners (published on June 22, 2013 by DrPhysicsA)
0:00 – 14:45 The general theory illustrated
14:48 – end The mathematics behind the theory, drawing on the discipline of what is now called Differential Geometry. The derivation of the tensors g_{uv}, R_{uv}, T_{uv} is discussed.