Math 101

THE FACTOR THEOREM

Taking a Closer Look at the Factor Theorem

Recall from page 100 of the Study Guide, Unit 3:

The **Factor Theorem** states the following:

For a polynomial $P$ of degree $n$ (a positive non-zero integer),

\[ P(a) = 0 \text{ where } a \text{ is any real number } \text{ if and only if } (x - a) \text{ is a factor of } P(x). \]

**Note that:** \((x - a)\) is a factor of \(P(x)\) is another way of saying \(P(x)\) is divisible by \((x - a)\).

The “if and only if” implication means that it works both ways.

The “if” implication:

**If** \(P(a) = 0\), where \(a\) is any real number, **Then** \((x - a)\) is a factor of \(P(x)\).

(First statement implies the second statement)

AND

The “only if” implication:

**If** \((x - a)\) is a factor of \(P(x)\), where \(a\) is any real number, **Then** \(P(a) = 0\).

(Second statement implies the first statement)

Examples of Using the Factor Theorem

Suppose \(P(x) = x^4 - x^2\). Then \(P\) is a polynomial of degree 4. In this case, \(n = 4 > 0\).

The “if” implication:

Observe that \(P(0) = 0^4 - 0^2 = 0\).

By the Factor Theorem, we can say that \((x - 0)\) or simply \(x\) is a factor of \(P(x) = x^4 - x^2\).

Observe that \(P(-1) = (-1)^4 - (-1)^2 = 1 - 1 = 0\).

By the Factor Theorem, we can say that \((x - (-1))\) or simply \((x +1)\) is a factor of \(P(x) = x^4 - x^2\).

The “only if” implication:

If we factor \(P(x) = x^4 - x^2\), we get \(P(x) = x^4 - x^2 = x^2(x^2 - 1) = x^2(x - 1)(x +1)\).

By the Factor Theorem, we can deduce that since

\(x = (x - 0), \ (x - 1), \ and \ (x +1) = (x - (-1))\) are all factors of \(P(x)\), then

\(P(0) = 0; \ P(1) = 0; \ and \ P(-1) = 0\)

– which is evident in this case but not necessarily in all cases.
Special Cases of the Factor Theorem

See page 103 of the Study Guide, Unit 3.

1. Let \( f(x) = x^n - a^n \) where \( n \) is a positive non-zero integer and \( a \) is any real number. Then \((x - a)\) is a factor of \( f(x) \).

Simply observe that \( f(x) = x^n - a^n \) is a special kind of polynomial of degree \( n \).
Since \( f(a) = a^n - a^n = 0 \), by the “if” implication of the Factor Theorem, we can deduce that \((x - a)\) is a factor of \( f(x) \).

2. Let \( f(x) = x^n + a^n \) where \( n \) is an ODD positive integer and \( a \) is any real number. Then \((x + a)\) is a factor of \( f(x) \).

Again \( f(x) = x^n + a^n \) is a special kind of polynomial of degree \( n \).
Since \( f(-a) = (-a)^n + a^n = -a + a = 0 \) (Here we have used the fact that \( n \) is an ODD positive integer), by the “if” implication of the Factor Theorem, we can deduce that \((x - (-a)) = (x + a)\) is a factor of \( f(x) \).

Examples Using the Special Cases of the Factor Theorem

Example 1:
Determine whether the number \( 5^6 - 8 \) is a prime number by using the Factor Theorem.

Solution:
Recall that a number \( p \) is a prime number if and only if its only factors are ± itself or ± 1.

We observe that the number \( 5^6 - 8 = (5^3)^2 - 2^3 \) has the form \( f(x) = x^n - a^n \) where \( x = 5^3 \), \( a = 2 \), and \( n = 3 \).

Now we apply the first special case of the Factor Theorem (listed above) to the polynomial \( f(x) = x^3 - 2^3 \).

We deduce that \((x - 2)\) is a factor of \( x^3 - 2^3 \).

Returning to our number, we recall that \( x = 5^2 \).
Therefore \((x - 2) = 5^2 - 2 = 23\) is a factor of \( x^3 - 2^3 = (5^3)^2 - 2^3 = 5^6 - 8 \).
Since \( 23 \neq \pm 1 \) and \( 23 \neq \pm (5^6 - 8) \), we conclude that \( 5^6 - 8 \) is not a prime number.
Example 2:
Given: \(5^p - 1\), where \(p\) is an even number (a positive integer divisible by 2).
Prove: that \(5^p - 1\) is also an even number.

HINT:
As \(p\) is representative of any even number, the proof must involve a general application of the Factor Theorem.

Solution:
Since the number \(p\) is divisible by 2, 2 is a factor of \(p\), implying that \(p = 2 \cdot n\) for some positive integer \(n\).

\(5^p - 1\) can then be rewritten as: \(5^p - 1 = 5^{2n} - 1 = (5^2)^n - 1 = (5^2)^n - 1^n\) and has the form
\(f(x) = x^n - a^n\) where \(x = 5^2\) and \(a = 1\).

(Here we used the fact that \(1^n = 1\) for any positive integer \(n\).)

If we apply the first special case of the Factor Theorem to the polynomial \(f(x) = x^n - a^n\),
we may deduce that \((x - a)\) is a factor of \(x^n - a^n\)
OR \((5^2 - 1)\) is a factor of \((5^2)^n - 1^n = 5^p - 1\),
showing that \(24\) is a factor of \(5^p - 1\).

If 24 is a factor of \(5^p - 1\), then 2 is also a factor of \(5^p - 1\).
Thus \(5^p - 1\) is an even number.

q.e.d. (Latin abbreviation for ‘has been proved’)

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**Example 3:**
Factor \( f(x) = x^5 + 32 \) into two factors.

**Solution:**
Observe that \( f(x) = x^5 + 32 = x^5 + 2^5 \). Here \( a = 2 \) and \( n = 5 \) (an **ODD** positive integer).

Applying the **second special case** of the Factor Theorem to the polynomial \( f(x) = x^5 + 2^5 \), we deduce that \((x + 2)\) is a factor of \( f(x) \).

We now divide \((x + 2)\) into \((x^5 + 32)\) using polynomial division.

\[
\begin{array}{c|ccccc}
    & x^4 & - 2 \cdot x^3 & + 4 \cdot x^2 & - 8 \cdot x & + 16 \\
\hline
x + 2 & x^5 & + 0 \cdot x^4 & + 0 \cdot x^3 & + 0 \cdot x^2 & + 0 \cdot x & + 32 \\
    & x^5 & + 2 \cdot x^4 & & & & \\
    & & - 2 \cdot x^4 & + 0 \cdot x^3 & & & \\
    & & - 2 \cdot x^4 & - 4 \cdot x^3 & & & \\
    & & & + 4 \cdot x^2 & + 0 \cdot x & & \\
    & & & + 4 \cdot x^2 & + 8 \cdot x^2 & & \\
    & & & & - 8 \cdot x^2 & + 0 \cdot x & \\
    & & & & - 8 \cdot x^2 & - 16 \cdot x & \\
    & & & & & + 16 \cdot x & + 32 \\
    & & & & & + 16 \cdot x & + 32 \\
    & & & & & & 0 \\
\end{array}
\]

Therefore, \( f(x) = x^5 + 32 = (x + 2) \cdot (x^4 - 2 \cdot x^3 + 4 \cdot x^2 - 8 \cdot x + 16) \).